Quantum and classical statistical mechanics of the sinh-Gordon equation

1986 J. Phys. A: Math. Gen. 19 L955

(http://iopscience.iop.org/0305-4470/19/16/002)

J. Phys. A: Math. Gen. 19 (1986) L955-L960. Printed in Great Britain

LETTER TO THE EDITOR

Quantum and classical statistical mechanics of the sinh-Gordon equation

R K Bullough†‡, D J Pilling§ and J Timonen‖

‡ Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
§ Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, UK
‖ Department of Physics, University of Jyväskylä, 40100 Jyväskylä, Finland

Received 27 June 1986

Abstract. We give two fundamental methods for quantum or classical free energies of integrable models. Periodic boundary conditions induce an integral equation for classically allowed momenta. Generalisations of the Bethe ansatz and a method of functional integration on the classical action in action-angle variables follow, giving identical (Bose-Fermi equivalent) results. For sinh-Gordon the Bose classical limits agree with the transfer integral method.

In covariant form the classical sinh-Gordon equation (sinh-G) is

$$\phi_{xx} - \phi_{tt} = m^2 \sinh \phi \tag{1}$$

where $\phi_{xx} \equiv \partial^2\phi/\partial x^2$, etc, and m is a mass ($\hbar = c = 1$). By the transformation $\phi \rightarrow -\mathrm{i}\phi$ it becomes the classical sine-Gordon equation (s-G). Both sinh-G and s-G are classical integrable models solvable by the inverse scattering (spectral transform (ST)) method [1]. Both are Hamiltonian systems and both are Liouville integrable with a complete, continuously infinite, set of commuting independent constants of the motion which form action variables [1-3]. However, on all of $-\infty < x < \infty$, the s-G has soliton (kink, antikink and breather) solutions: the sinh-G has no soliton solutions.

The normally ordered quantum s-G at zero temperature $\beta^{-1} \equiv T = 0$ has been solved for its eigenspectrum and eigenstates by the methods of the Bethe ansatz (BA) and quantum inverse method (QIM) [4, 5] (see also [2]). The quantum statistical mechanics of the s-G is available as a system of coupled integral equations [6] which have now been partially solved in the classical limit [7]¶. No such results are available for quantum or classical sinh-G at $T > 0$ or $T = 0$. But although it is the statistical mechanics of s-G which is relevant to experiment so far [9], the very explicit connection of sinh-G to s-G (see below), and its relative simplicity, makes it ideal for study. In this letter we report two new, different and fundamental methods of calculating the quantum and classical free energies of a whole class of integrable models; we use sinh-G as one example of these. In another communication [8] we will generalise both methods to the still larger class which includes s-G.

† On leave from UMIST, PO Box 88, Manchester M60 1QD, UK.
¶ Fowler and his colleagues now report what we show is a *complete* solution in the classical limit [8]. We are grateful to Professor Fowler for conveying his results to us by telephone.

The two fundamental methods we report are a method of functional integration and a method of integral equations generalising the BA method—which, however, *makes no ansatz whatsoever.* The *classical* free energy for sinh-G can be calculated from the functional integral

$$Z = \int \mathscr{D}\Pi \mathscr{D}\phi \exp(-\beta H[\phi]) \tag{2}$$

by a method already available: the transfer integral method (TIM) [2, 10]. For sinh-G the classical Hamiltonian $H[\phi]$ is

$$H[\phi] = \gamma_0^{-1} \int [\tfrac{1}{2}\gamma_0^2\Pi^2 + \tfrac{1}{2}\phi_x^2 + m^2(\cosh\phi - 1)]\, dx \tag{3}$$

with Poisson bracket $\{\Pi, \phi\} = \delta(x - x')$; γ_0 is a dimensionless coupling constant. We find a result by the TIM which coincides with that we give (in (13) below) as the joint result in the classical limit of the two new quantum theoretical methods reported in this letter. Thus the TIM serves to check these methods. Connections with s-G are then that $H[\phi]$ for s-G follows from (3) by $\phi \to -i\phi$, $\gamma_0 \to -\gamma_0$ and the classical free energy of s-G can be found from (13) by analytical continuation in γ_0.

Both the methods we report in this letter exploit the fact that under the ST a classical integrable system like sinh-G undergoes a *canonical* transformation to action-angle variables [1-3]. Thus (3) transforms to

$$H[p] = \int_{-\infty}^{\infty} \omega(k) P(k)\, dk \tag{4}$$

in which $\omega(k) \equiv (m^2 + k^2)^{1/2}$. The $P(k)$, $0 \leq P(k) < \infty$, are action variables and the $Q(k)$, $0 \leq Q < 2\pi$, are angle variables. Also $\{P, Q\} = \delta(k - k')$. In these variables the *quantum* functional integral Z for sinh-G is [2]

$$Z = \int \mathscr{D}\mu \exp(S[p]) \tag{5}$$

in which $S[p]$ is the Wick rotated *classical* action

$$S[p] = \hbar^{-1} \int_0^{\hbar\beta} d\tau \left(i \int_{-\infty}^{\infty} dk\, P(k) Q(k)_\tau - H[p] \right) \tag{6}$$

and $\mathscr{D}\mu \propto \mathscr{D}P\mathscr{D}Q$: the classical limit, which when correctly handled is equivalent to (2), sets $\hbar \to 0$ and so replaces $S[p]$ by $-\beta H[p]$ [2].

The Hamiltonian (4) for sinh-G is particularly simple since it represents a bunch of harmonic oscillators [2]. However, the action variables are determined by the spectral data of the classical ST method [1-3] and it is the quantum operator forms of these spectral data which underlie both the QIM [4, 5] and the BA method [4, 5]. Therefore it becomes important to draw the connection between the BA, QIM and methods of functional integration involving the classical action $S[p]$ and the classical spectral data. We do this for the large class of integrable models with Hamiltonians like (4) by doing it explicitly for sinh-G in this letter.

The two methods of calculation we report are functional integration using action-angle variables exemplified by (5) and (6) and a method which follows, but generalises, the method of BA pioneered by Yang and Yang [11]. This concerned the quantum statistical problem of $N+1$ bosons on a line with repulsive δ-function interactions of strength $c>0$: as is well known [2, 5] this is the case of the quantised normally ordered repulsive non-linear Schrödinger equation (NLS). We show how our results for sinh-G correspond to those for quantum NLS: the connection in the classical limit is also drawn. The essential point is that the classical NLS has action-angle variables P, Q and Hamiltonian (4) with $\omega(k)=k^2$.

Both methods depend on the need to impose periodic boundary conditions, of period $L<\infty$, on the problem. From this a proper thermodynamic limit is achieved at finite density for $L\to\infty$. Without reinterpretation (4) is not at finite density: it is derived [1-3] for ϕ on the real line with decaying boundary conditions at infinity and action-angle variables for sinh-G (or s-G) have not been available for periodic boundary conditions†. A main point of this letter is to show how periodic boundary conditions are imposed on both methods so that, e.g., the functional integral (5) *and* its classical limit are evaluated in proper limit at finite density.

Here it is convenient to report the generalisation of the BA method first and the functional integral method second. We have used classical Floquet theory [2, 12] (see also the references in [2, 4]) to show that under periodic boundary conditions of large period L the Hamiltonian (4) for classical sinh-G is not free but is constrained by the condition for allowed wavevectors $\tilde{k}_n$ (n, m integers)

$$L\tilde{k}_n=2\pi n-\sum_{m\neq n}\Delta(\tilde{k}_n,\tilde{k}_m)P_m+\mathrm{O}(L^{-1}). \tag{7}$$

Under these conditions the analysis of the relevant lattice† of spacing a and period $(N+1)a=L$ (N is even for convenience) shows that (4) is replaced by $(-\frac{1}{2}N\leqslant n\leqslant\frac{1}{2}N)$

$$H[p]=\sum\omega(\tilde{k}_n)P_n+\mathrm{O}(L^{-1}) \tag{8}$$

and $P_m\leftrightarrow P(\tilde{k}_m)2\pi L^{-1}$. To $\mathrm{O}(L^{-1})$ we can use the analytical properties of the transmission coefficient $a(\zeta)$ [1-3] associated with the ST on the real line in classical form to show that for sinh-G $\Delta(k,k')$ is given by

$$\Delta_{\mathrm{b}}(k,k')=-\tfrac{1}{4}m^2\gamma_0[k\omega(k')-k'\omega(k)]^{-1} \tag{9}$$

which is recognisable as a classical propagator for phonons of arbitrary amplitude [2]. Moreover, for the quantum case we have calculated the phase shift Δ to be‡ $\Delta(k,k')=-2\tan^{-1}\{m^2\sin(\frac{1}{8}\gamma_0'')[k\omega(k')-k'\omega(k)]^{-1}\}$ in which $\gamma_0''\equiv\gamma_0(1+\gamma_0/8\pi)^{-1}$ and this two-body S-matrix phase shift coincides with s-G for $\gamma_0\to-\gamma_0$ [4]. It now becomes important to distinguish 'Bose' shifts Δ_{b} and 'Fermi' shifts Δ_{f} [2, 5]. For, in contrast with previous work [5, 11], (9) shows we must use the Bose form $\Delta\equiv\Delta_{\mathrm{b}}(k,k')=\Delta_{\mathrm{f}}-2\pi\theta(k'-k)$ in which $\theta(k)=0(k<0)=1(k>0)$. Thus Δ_{f} is the smooth branch of Δ in which $\Delta\to-2\pi(k\to-\infty)$ and $\to0(k\to+\infty)$ (cf [5]), while $\Delta\equiv\Delta_{\mathrm{b}}\to0(k\to\pm\infty)$ reducing to (9) in the 'classical limit' $\gamma_0\to0$.

For the functional integral method (see below) P_m and Q_m $(0\leqslant Q_m<2\pi)$ are canonical: $\{P_m,Q_m\}=\delta_{mn}$. But one can also see, and show, that P_m plays the role of a particle density. To ensure a finite (boson) density $\rho(k)$ for $L\to\infty$, let $L^{-1}P_m\to L^{-1}(2\pi L^{-1}P(\tilde{k}_n))\leftrightarrow\rho(k)\,\mathrm{d}k$ for $L\to\infty$. Let $\tilde{k}_nL\to h(k)L$ and $k_n\equiv2\pi nL^{-1}\to k$ for $L\to\infty$. The density of states $f(k)\equiv(2\pi)^{-1}\,\mathrm{d}h/\mathrm{d}k$, so for $L\to\infty$ (7) means

$$2\pi f(k)=1-\int_{-\infty}^{\infty}(\mathrm{d}\Delta_{\mathrm{b}}/\mathrm{d}k)\rho(k')\,\mathrm{d}k'. \tag{10}$$

† We shall report elsewhere variables of this type for both a lattice sinh-G and a lattice s-G under periodic boundary conditions.

‡ To this end we have given a BA-QIM analysis for quantum sinh-G which is to be reported elsewhere.

Following Yang and Yang [11], but using Bose rather than Fermi particles, the entropy SL^{-1} per unit length is $SL^{-1}=\int_{-\infty}^{\infty}[(f+\rho)\ln(f+\rho)-f\ln f-\rho\ln\rho]\,dk$ as $L\to\infty$. From (4) and (8) the energy $EL^{-1}=\int_{-\infty}^{\infty}\omega(k)\rho(k)\,dk$. Note how $\omega(k)$ replaces $\omega(\tilde{k})$ since, by (7), terms correcting this are $O(L^{-1})$. We now minimise the free energy $FL^{-1}=(E-\beta^{-1}S)L^{-1}$ with respect to ρ. This reduces (10) to an integral equation. Then, if we conveniently define a boson energy $\varepsilon(k)$ by $f\rho^{-1}+1\equiv\exp(\beta\varepsilon(k))$, this becomes

$$\varepsilon(k)=\omega(k)+(2\pi\beta)^{-1}\int_{-\infty}^{\infty}(d\Delta_b(k,k')/dk)\ln[1-\exp(-\beta\varepsilon(k'))]\,dk' \tag{11a}$$

while the free energy is

$$FL^{-1}=(2\pi\beta)^{-1}\int_{-\infty}^{\infty}\ln[1-\exp(-\beta\varepsilon(k))]\,dk. \tag{11b}$$

We note that if $\Delta_b=-2\tan^{-1}[c(k-k')^{-1}]$ and Δ_f is the branch of the $\tan^{-1}$ which $\to-2\pi$ $(k\to-\infty)$ and $\to 0$ $(k\to+\infty)$ (the appropriate branch of the S-matrix phase shift for quantum NLS in fermion description [5, 11]) equations (11) are *boson* forms of the Yang and Yang result [11] providing $\omega(k)=k^2$†. Indeed, quite generally, i.e. for any $\omega(k)$‡ and Δ_b, since $d\Delta_b/dk=d\Delta_f/dk-2\pi\delta(k)$, one can usefully define *fermion* energies $\bar{\varepsilon}(k)$ by $\ln[1+\exp(-\beta\bar{\varepsilon}(k))]=-\ln[1-\exp(-\beta\varepsilon(k))]$. Also with $\bar{n}\equiv NL^{-1}=\int_{-\infty}^{\infty}\rho(k)\,dk$, one can minimise the negative pressure $-p\equiv FL^{-1}-\mu NL^{-1}$ instead of FL^{-1} to find from (11) the exact forms found [11]

$$\bar{\varepsilon}(k)=\omega(k)-\mu+(2\pi\beta)^{-1}\int_{-\infty}^{\infty}d\Delta_f(k,k')/dk'\ln[1+\exp(-\beta\bar{\varepsilon}(k'))]\,dk' \tag{12}$$

with $FL^{-1}=\mu\bar{n}-(2\pi\beta)^{-1}\int_{-\infty}^{\infty}\ln[1+\exp(-\beta\bar{\varepsilon}(k))]\,dk$ and μ the chemical potential. New features from [11] are therefore the calculation for sinh-G and the Bose description. But the real point is the reliance on (4) and the Floquet results (7) and (8); evidently the argument applies to all $(1+1)$-dimensional integrable models without soliton solutions‡. In particular, for sinh-G, Δ_f is the smooth branch of the quantum Δ and $\omega(k)=(m^2+k^2)^{1/2}$; for NLS, Δ_f is the smooth branch of $-2\tan^{-1}[c(k-k')^{-1}]$ and $\omega(k)=k^2$.

The Bose-Fermi equivalence of the two quantum forms (11) and (12) has many interesting facets—particularly for quantum NLS [5] and quantum sinh-G at $T>0$ and $T=0$. The quantum eigenspectra follow at $T=0$ [5]. Here we merely note that the *Bose* form (11) has the easy classical limit in which $\ln(\beta\varepsilon(k))$ replaces $\ln[1-\exp(-\beta\varepsilon(k))]$. With Δ_b given by (9) for sinh-G, iteration yields

$$FL^{-1}=m\beta^{-1}[\tfrac{1}{4}(M\beta)^{-1}-\tfrac{1}{8}(M\beta)^{-2}+\tfrac{3}{16}(M\beta)^{-3}-\tfrac{53}{128}(M\beta)^{-4}+\cdots]+F_{KG} \tag{13}$$

where $F_{KG}\equiv\beta^{-1}a^{-1}(\ln\beta a^{-1}+\tfrac{1}{2}ma)$ and $M\equiv 8m\gamma_0^{-1}$. For F_{KG} in this form we must use the dispersion relation $\omega(k)$ for a linear Klein-Gordon lattice as the lattice theory shows: F_{KG} *must* depend on the lattice spacing a because of the classical limit. The strictly asymptotic series (13) is exactly what we derive by using the TIM on (2) with Hamiltonian (3). By analytical continuation in γ_0 (so that $\gamma_0\to-\gamma_0$) (13) becomes the free energy of s-G which contains additional soliton contributions [14]. In principle an expression like (13) is similarly found from (11) in the classical limit for the NLS, but iteration is not possible in this case.

† Wadati [13] reports the particular calculation for NLS in this form.

‡ The linear dispersion relation $\omega(k)$ wholly determines the classical integrable model [1].

We are now in a position to illustrate the second, functional integration, method. With (5) in the classical limit as our first example our procedure is to discretise the classical action $-\beta H[p]$ so that

$$Z=\lim_{N\to\infty}(2\pi)^{-(N+1)}\int\cdots\int\prod_{n=-\frac{1}{2}N}^{+\frac{1}{2}N}\mathrm{d}P_n\,\mathrm{d}Q_n\exp\left(-\beta\sum_{n=-\frac{1}{2}N}^{+\frac{1}{2}N}\omega(\tilde{k}_n)P_n\right) \tag{14}$$

with $(N+1)a=L$. The P_n are now interpreted as action variables with canonical Q_n: the normalisation corresponds to $h=2\pi$ for each oscillator mode label n. The fundamental point now is that the constraint on the allowed modes $\tilde{k}_n$ is still given by the periodicity condition (7) (with $\Delta=\Delta_b$ from (9)). One must now use the fact that the $\omega(\tilde{k}_n)$ depend on the P_n. Thus, in this case of classical limit, we find after some work

$$\begin{aligned}FL^{-1}&=(2\pi\beta)^{-1}\int_{-\infty}^{\infty}\mathrm{d}k\ln(\beta\omega(k))-(2\pi\beta)^{-2}\int_{-\infty}^{\infty}\mathrm{d}q[\omega(q)]^{-1}\int_{-\infty}^{\infty}\mathrm{d}k\\&\quad\times\Delta_b(k,q)\,\mathrm{d}(\ln\omega(k))/\mathrm{d}k+(2\pi\beta)^{-3}\int_{-\infty}^{\infty}\mathrm{d}p[\omega(p)]^{-1}\int_{-\infty}^{\infty}\mathrm{d}q\\&\quad\times\Delta_b(q,p)\frac{\partial}{\partial q}\left([\omega(q)]^{-1}\int_{-\infty}^{\infty}\mathrm{d}k\,\Delta_b(k,q)\,\mathrm{d}(\ln\omega(k)/\mathrm{d}k)\right)\\&\quad-\tfrac{1}{2}(2\pi\beta)^{-3}\int_{-\infty}^{\infty}\mathrm{d}q[\omega(q)]^{-2}\left(\int_{-\infty}^{\infty}\mathrm{d}k\,\Delta_b(k,q)\,\mathrm{d}(\ln\omega(k))/\mathrm{d}k\right)^2+\cdots.\end{aligned} \tag{15}$$

One can then see this is precisely the iteration of (11) in the classical limit! Thus the periodic boundary condition (7) in its classical form ensures that the non-linearity ($\propto\gamma_0$) emerges in the expressions for the free energy. It is easy to generalise the analysis through (15) to the quantum case by using the quantum forms (5) with (6) for Z. It is convenient here to impose the oscillator WKB quantisation conditions $\oint P_n\,\mathrm{d}Q_n=2\pi n$, so $P_n=n$, a boson number. This way we readily regain the Bose forms (11) for the quantum sinh-G exactly. The Fermi forms follow as before.

Evidently the crucial content of the argument is the constraint (7) imposed by periodic boundary conditions. Obviously it generalises the BA condition [5]: indeed, if P_n is a fermion number 0, 1 and Δ is Δ_f, (7) is exactly the allowed mode condition [5]. On the other hand, the P_n for the Bose form are boson numbers and in the classical analysis of the functional integral though (15) the P_n are the usual action variables, while the alternative analysis through to (11) shows that the classical limit can be obtained directly by interpreting both the $f\,\mathrm{d}k$ and the $\rho\,\mathrm{d}k$ (which corresponds to P_n) classically in terms of Maxwell-Boltzmann statistics. Thus the periodicity condition (7) seems to be a major generalisation of the periodicity conditions of BA and QIM.

Every point of this analysis has its generalisation to the quantum and classical statistical mechanics of the sine-Gordon (s-G) equation. We shall report all of this in another communication [8].

One of us (RKB) is grateful for the support of the 'integrable models' programme at the Institute for Theoretical Physics, UCSB. Both RKB and DJP are grateful to the UK SERC for support.

Note added in proof. The work of Fowler and colleagues on the classical limit of the BA thermodynamics for quantum s-G is reported (Chen, Johnson and Fowler 1986 *Phys. Rev. Lett.* **56** 904 (erratum **56** 1427)) for the case $8\pi\gamma_0^{-1}=n$, an integer. The different calculation for $8\pi\gamma_0^{-1}=n+\varepsilon$ (ε an infinitesimal, >0) is given by ourselves (Timonen *et al* 1986 *Phys. Rev.* B to be published). Results coincide with the analytic continuation of the result (13) of this letter as found in [8].

References

[1] Bullough R K and Caudrey P J (ed) 1980 *Solitons* (*Topics in Current Physics* **17**) (Berlin: Springer)
Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)

[2] Bullough R K, Pilling D J and Timonen J 1985 *Nonlinear Phenomena in Physics* ed F Claro (Berlin: Springer) pp 70-102, 103-28

[3] Takhtadzhyan L A and Faddeev L D 1974 *Teor. Mat. Fiz.* **21** 160
Dodd R K and Bullough R K 1979 *Phys. Scr.* **20** 514

[4] Skylanin E K, Takhtadzhyan L A and Faddeev L D 1979 *Teor. Mat. Fiz.* **40** 194
Faddeev L D 1983 *Proc. Ecole d'Eté de Physique Theorique, Les Houches, 1982* ed R Stora and J B Zuber (Amsterdam: North-Holland) pp 561-608

[5] Thacker H B 1981 *Rev. Mod. Phys.* **53** 253

[6] Takahashi M and Suzuki M 1972 *Prog. Theor. Phys.* **48** 2187

[7] Fowler M 1981 *Phys. Rev.* B **26** 2514
Maki K 1985 *Phys. Rev.* B **32** 3075

[8] Timonen J, Stirland M, Pilling D J, Yi Cheng and Bullough R K 1986 *Phys. Rev. Lett.* **56** 2233
Timonen J, Bullough R K and Pilling D J to be published

[9] Lovesey S W, Balucani U, Borsa F and Tognetti V (ed) *Magnetic Excitations and Fluctuations* (Berlin: Springer)

[10] Scalapino D J, Sears M and Ferrell R S 1972 *Phys. Rev.* B **6** 3409

[11] Yang C N and Yang C P 1969 *J. Math. Phys.* **10** 1115

[12] Flaschka H and McLaughlin D W 1970 *Prog. Theor. Phys.* **55** 438

[13] Wadati M 1985 *J. Phys. Soc. Japan* **54** 3727

[14] Bullough R K, Pilling D J, Stirland M and Timonen J 1985 *Physica* **16D** 368
Stone M and Reeve J 1978 *Phys. Rev.* D **18** 4746